COMBINATORICS

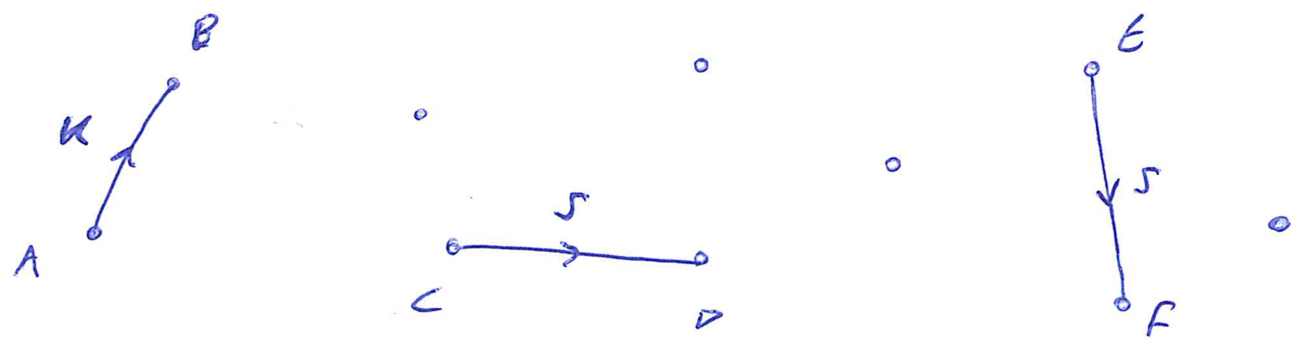
THE "KNIGHTS AND SPIES" PROBLEM:

100 people are living on an island. Each is either a Knight or a Spy. Knights always tell the truth, but Spies can either lie or tell the truth. There are more Knights than Spies on the island.

You arrive on the island, and you are allowed to ask questions of the form "is X a Knight?" or "is X a Spy?"

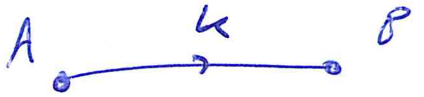
Show that it is possible, using 150 questions, to determine who are the Knights and who are the Spies.

Observations



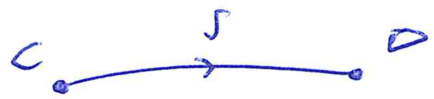
Draw people as dots.

Ask A about B, get directed edge from A to B labelled with answer (K or S).



Possibilities:

(A, B)
(K, K)
(S, K)
(S, S)



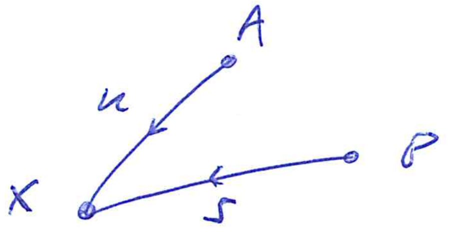
Possibilities:

(C, D)
(K, S)
(S, K)
(S, S)

1. Disconnected questioning not a good strategy.

In fact, a good strategy would be to ask repeatedly about one person (call this person X).

e.g.



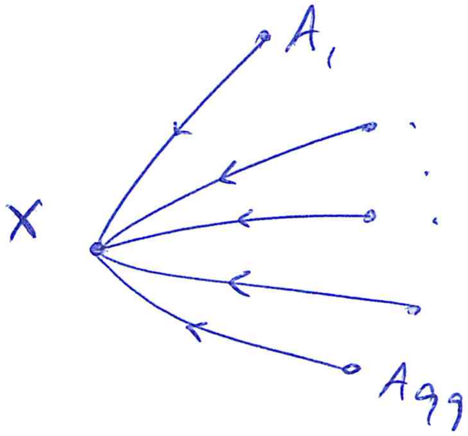
In this case, either A or B (or both) is a Spy.

2. If we can identify a Knight, then he (she) can help us to identify everyone else!

3. The entire group (population) has a "Truthful Majority" property — there are more Knights than Spies.

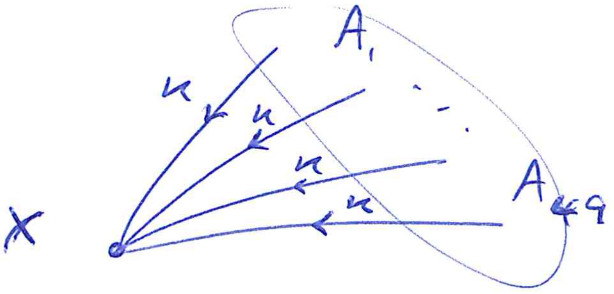
So if we ask <u>everyone</u> (99 people) about X, we can trust the majority answer.

i.e.

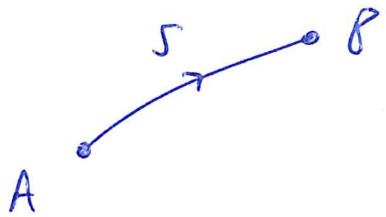


If at least 50 of $\{A_i\}$ say "Knight", then X is a Knight. [same for Spy].

4. If (asking about X) at some stage,
~~[illegible]~~ 49 people say that X is a knight, then
X is a knight.



5.

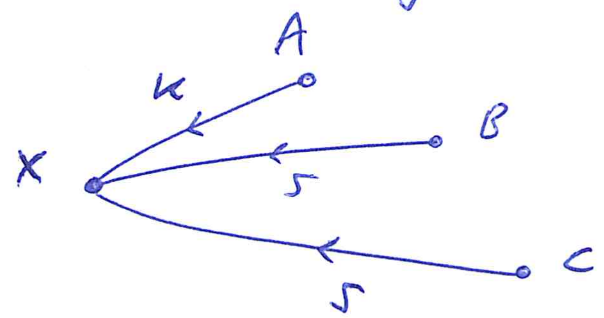


If A calls B a spy, either

(i) A is a knight (A = k) $\Rightarrow$ B = S
or
(ii) A is a spy. (A = S)

$\Rightarrow$ Conclude that at least one of {A, B} is a spy.

6. Another possibility:



If $X = K$, then $\{B, C\}$ are SPIES

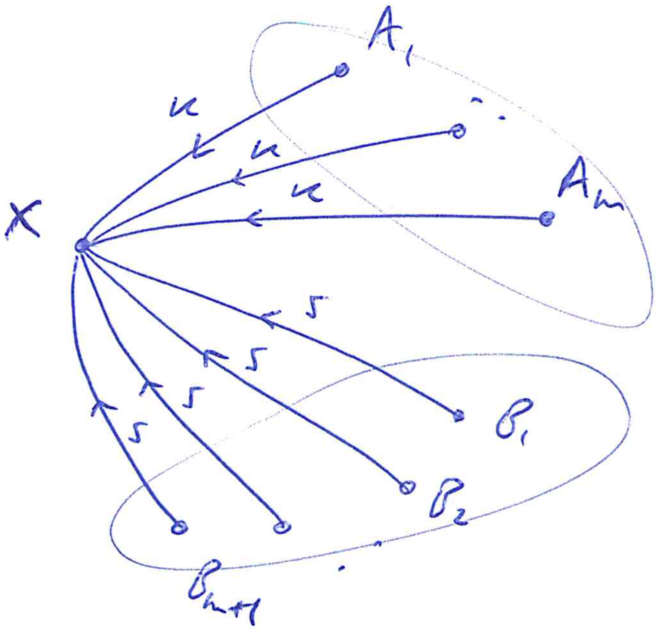
If $X = S$, then $\{A, X\}$ are SPIES.

Either way, ≥ 2 of the 4 people are SPIES.

<u>THIS LEADS TO OUR KEY TRICK</u>:

Keep asking about 1 person (X). As soon as we get more Spy answers than Knight answers, <u>cut off</u> the group.

In general, for $m \geq 0$, we have in this case



$(2m+2)$ people in group cut off.

If $X = K$, then $\{B_1, \ldots, B_{m+1}\}$ are all SPIES

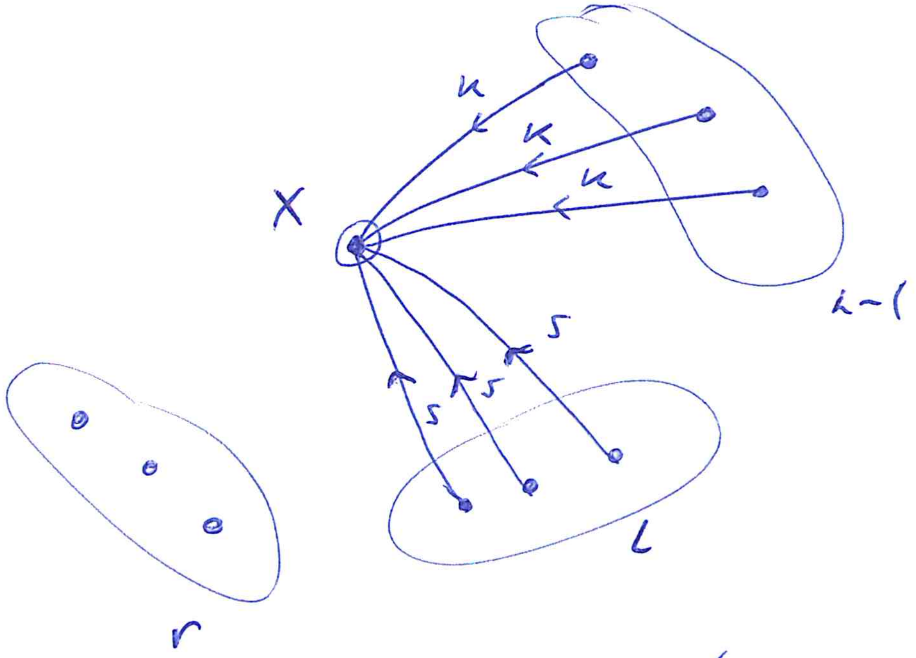
If $X = S$, then $\{X, A_1, \ldots A_m\}$ are all SPIES

Either way, at least $\frac{1}{2}$ of the group are SPIES ($m+1$ of $2m+2$ people).

After cutting off the group, the remaining group still has the truthful majority property.

Keep doing this until we eventually get $(n-1)$ Knight answers ~~...~~ among the $2n$ remaining people. [This MUST eventually happen!]

2n people remaining



Here $(n-1)+1+r+l = 2n$

$\Rightarrow \boxed{r+l = n}$

If $X = S$, we'd have identified n spies at of $2n$ $\Rightarrow$ Impossible!

$\Rightarrow$ X is a <u>KNIGHT</u>.

Used (for this group) $n-1+l$ questions.

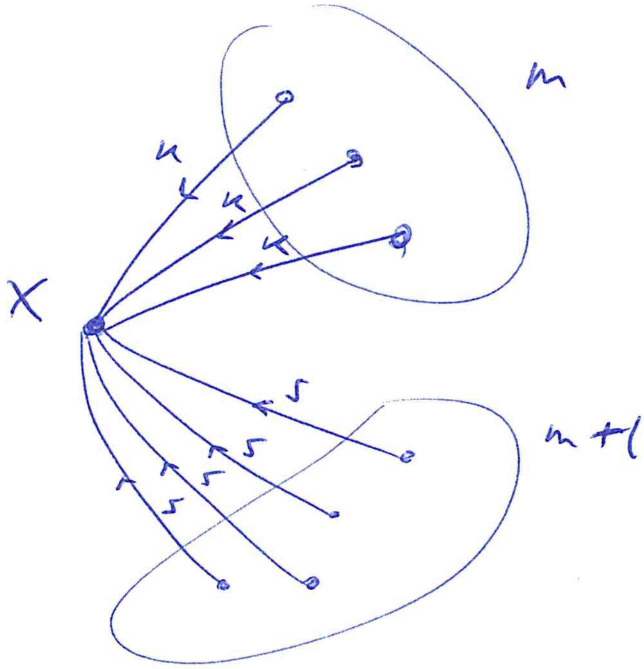
Need another $\underline{n-1+r}$ questions

In total, need $(2n-2)+(l+r)$

$= 3n-2$ questions

to identify this group of $2n$.

Now, finally use our knight to resolve identities in groups we cut off earlier. Each one looks like:



Used $(2m+1)$ questions

Knight asks X about identity.

If $X = k$, need $(m+1)$ questions

If $X = s$, need $(m+2)$ questions

In total, we need $\leq (2m+1) + (m+2)$

$= 3m+3$ questions

$\Rightarrow$ $3m+3$ Qs to identify $(2m+2)$ people.

To summarize, we require 148 questions to identify the 150 islanders [$3n-2$ for $2n$ people]. QED